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Distributionally robust ramp metering under traffic demand uncertainty

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ABSTRACT

This study presents a distributionally robust optimization model to address the ramp metering problem with uncertain traffic demand flows. The aim of this model is to minimize the total travel delay of the system based on the macroscopic cell transmission model (CTM) of traffic flow. In our model, the only required data is the partial distributional information of stochastic demand flows. Using the Worst-Case Conditional Value-at-Risk (WCVaR) constraints to approximate the distributionally robust chance constraints, the proposed problem can be conservatively approximated as a semidefinite programming (SDP), which is computationally efficient. The performances of our proposed model are illustrated by practical applications. Experimental results show that the distributionally robust control strategy can achieve reliable performances over a range of uncertain scenarios.

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Cell transmission model; distributionally robust chance constraint; ramp metering; semidefinite programming; Worst-Case Conditional Value-at-Risk

1. Introduction

The congestion of traffic has a significant effect on various social issues, such as public health, safety, fuel consumption, environment and security. The social costs of traffic congestion across Australian capital cities are about 20.4 billion in 2020 based on the estimation of the Bureau of Infrastructure, Transport and Regional Economics (BITRE) (Systematics 2005). Over the past several decades, a set of practical measures and control strategies, including public transportation services, infrastructure expansions, and several operational enhancements known collectively as Intelligent Transportation Systems (ITS) have been proposed to improve freeway operations. One of the operational strategies for improving the freeway operation is ramp metering, which limits the incoming flow from on-ramps to the freeway.

The seminal research of optimization-based ramp metering can be traced back to the work (Wattleworth 1963), where a static model of traffic behaviour was used to formulate the problem. This model was subsequently investigated and extended by Yuan and Kreer (1971); Wang and May (1973); Iida et al. (1990). One of the most widely adopted classes of models in the freeway control design is the macroscopic models, including the first order models (Cell transmission model (CTM)) and the second order models (Metanet). The CTM model initially proposed by Daganzo (1994, 1995) can be regarded as a first-order Godunov approximation of the continuous Lighthill-Whitham-Richards-model (LWR) (Lighthill and Whitham 1955; Richards 1956), and Metanet was proposed by Messmer and Papageorgiou (1990). Particularly, Papageorgiou et al. (2003) and Papageorgiou and Kotsialos (2002) concluded that freeway ramp metering is a useful and effective tool to improve traffic flows on congestion-prone

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freeways. Kotsialos and Papageorgiou (2004) proposed a model-predictive framework for coordinated ramp metering rooted in the METANET model and formulated the considered problem as a nonlinear optimization problem. Based on the asymmetric cell transmission model, Gomes and Horowitz (2006) considered an on-ramp metering problem, where both free flow and congested conditions can be captured by the problem formulation. Gomes et al. (2008) further provided a theoretical analysis to study the behaviour of the CTM model of a freeway with steady demand. To analyse traffic flow density on freeway sections with random demand and supply, Sumalee et al. (2011) developed a stochastic cell transmission model by characterizing the probability distributions of occurrence of each mode.

Recently, Chow and Li (2014) proposed a robust optimization model of dynamic motorway traffic flow to optimize the total travel delay of the system with random traffic flow demands as well as set-valued fundamental diagrams (Kurzhanskiy and Varaiya 2012). The problem was reformulated as a minimization and maximization problem when an ellipsoidal likelihood set was considered. Roncoli, Papageorgiou, and Papamichail (2015a) developed a novel first-order multi-lane macroscopic traffic flow model for motorways to consider lane changing and capacity drop via appropriate procedures for computing lateral and longitudinal flows. Based on the work (Roncoli, Papageorgiou, and Papamichail 2015a), Roncoli, Papageorgiou, and Papamichail (2015b) proposed a linearly constrained optimal control model by permitting the deployment of lane changing control, variable speed limits, and ramp metering. Han et al. (2015) proposed a general first-order traffic flow model to simulate the capacity drop at the on-ramp bottleneck and lane drop bottleneck. On this basis, a linear quadratic model for predictive control strategy was proposed to realize the integration of dynamic path guidance and ramp metering. Furthermore, Han et al. (2017) considered the propagation of shockwave on the freeway network, and modified the supply function that depends on the density difference between cell i and its upstream cell $i-1$, where both cells are congested. In addition, on the arterial links where shockwave is generated during each cycle, if the upstream is in a free-flow condition, as in the modification (Han et al. 2017), the demand function of the target cell will have the same structure as the traditional CTM, and it will overestimate the actual discharge rate in the target cell. Under the assumption that equipped vehicles can bidirectionally communicate with the infrastructures, a novel feedback based integrated control strategy was proposed by Tajdari, Roncoli, and Papageorgiou (2020) to implement ramp metering and lane-changing control. By adjusting the adaptive cruise control (ACC) settings of equipped and connected vehicles in real time on the basis of the current traffic conditions, a simple and effective ACC-based control strategy is proposed by Spiliopoulo et al. (2018), where this control strategy relies only on real-time information about the current traffic conditions (no network topology information is required). Kontorinaki, Karafyllis, and Papageorgiou (2017) proposed a local and coordinated ramp metering strategy based on the nonlinear adaptive control scheme, which consists of a nominal feedback law and a nonlinear observer aimed at estimating some unknown system variables.

However, solutions obtained from the deterministic optimization models (DOM) and robust optimization models (ROM) are overly conservative. An adjustable robust optimization approach has been developed to alleviate the conservatism (Ben-Tal et al. 2004). Based on the work, Zymler, Kuhn, and Rustem (2013) proposed a novel method to approximate the distributionally robust individual and joint chance constraints with the first- and second-order moments and the support of the uncertainties of parameters. The approach is effective and outperforms the approximation proposed by Chen et al. (2010) and Bonferroni approximation. Although a number of researches have studied the robust solutions of ramp metering optimization, little has been done on problems where only partial information (such as mean and variance) of traffic parameters is available. Since the information provided by loop detectors may be incomplete in practice, using exact information of traffic parameters to study the traffic problem is often not possible in practice. It is an interesting point that we can conduct our research. The major contribution of this paper can be concluded as follows:

Firstly, we propose a distributionally robust chance constrained optimization model (DRCCOM) via ramp metering that incorporates the uncertain flow demands in the triangular fundamental diagrams. The partial distributional information of stochastic demand flows is given.

Secondly, the Worst-Case Conditional Value-at-Risk (WCVaR) constraints are adopted to approximate the distributionally robust chance constraints in our model, and then the approach (Zymler, Kuhn, and Rustem 2013) was applied to approximate the WCVaR constraints by the semidefinite programming (SDP) constraints. The approximation is exact and computationally efficient for distributionally robust individual chance constraints if the constraint functions are concave.

Finally, numerical results show that our model is efficient and outperforms deterministic optimization and robust optimization for minimizing the total delay of the freeway system.

The rest of the paper is organized as follows: Section 2 models the traffic flow dynamics. In section 3, we introduce the considered optimization problem rooted in the deterministic cell transmission model. The approximation of distributionally robust chance constraints will be presented in Section 4. In Section 5, the performances of various control strategies are illustrated and compared using practical examples. Finally, Section 6 gives some conclusions.

2. Modelling traffic flow dynamics

Lighthill and Whitham (1955) and Richards (1956) proposed the simplest continuous macroscopic model which is known as the kinematic wave model and is also called LWR model. The model is given by a single partial differential equation based on the conservation of vehicles. To extend and discretize the LWR model, a lot of work has been done. The CTM proposed by Daganzo (1994) is one of the most widely utilized discrete models. Due to the popularity and credibility of CTM, we utilize CTM to model the traffic flow dynamics in this paper.

In the formulation of CTM, a freeway is divided into I subsections or cells (see Figure 1). Each cell has an external incoming flow $r_{i,t}$ from an on-ramp i to the freeway and an external outgoing flow $s_{i,t}$ from the freeway to an off-ramp i at time step t, and the flow $f_{i,t}$ and density $\rho_{i,t}$ in each cell i at each time step t can be used to characterize the traffic flow dynamics. Let $f_{i-1,t}$ denote the traffic inflow to downstream cell i at each simulation time step t and, hence $f_{i,t}$ (inflow to downstream cell $i+1$) denotes the traffic outflow from cell i at the same simulation time step t. Based on the conservation equation, the evolution of density in cell i is described as follows:

$$\rho_{i,t+1} = \rho_{i,t} + \frac{\Delta t}{\Delta x_i}(f_{i-1,t} - f_{i,t} + r_{i,t} - s_{i,t}) \tag{1}$$

where Δx_i and Δt is the length of the cell i and the size of the simulation time step t, respectively. Depending on the network topology, some terms of Equation (1) may not be present. In particular, the inflow $f_{0,t}$ does not exist for the first cell of the network, the inflow $r_{i,t}$ does not exist for the cell without an on-ramp, while the outflow $s_{i,t}$ exists only for the cell with an off-ramp. It is noted that the time step Δt is defined such that $\Delta t \leq \min_i(\Delta x_i/v_i)$, which is the smallest ratio of the cell length Δx_i to the corresponding free flow velocity v_i on the freeway. The condition is used in traffic flow modelling to guarantee the numerical stability and nonnegativity of traffic quantities by limiting the distance travelled by vehicles in one simulation time step to no more than the length of the cell.

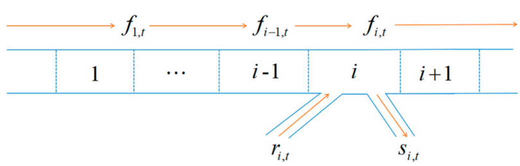

Figure 1. Schematic diagram.

In the case of a given cell density, the outflow from the cell i during the time step t is controlled by the min(.) function as follows:

$$f_{i,t} = \min\{v_i\rho_{i,t}, C_i, C_{i+1}, w_{i+1}(\rho_{\max,i+1} - \rho_{i+1,t})\} \tag{2}$$

where C_i represents the capacity flow of cell i and C_{i+1} denotes the capacity flow of cell $i+1$. Due to the heterogeneous segments with different capacities at different locations, we consider both capacity flows in adjacent cells. Furthermore, v_i represents the free flow velocity of cell i, w_{i+1} denotes the backward wave speed of cell $i+1$ and can be obtained from the equality $w_{i+1} = C_{i+1}/(\rho_{\max,i+1} - \rho^c_{i+1})$, where $\rho_{\max,i+1}$ is the jam density and ρ^c_{i+1} corresponds to the critical density and can be derived as $\rho^c_{i+1} = C_{i+1}/v_{i+1}$. Define $y^d_{i,t} = \min\{v_i\rho_{i,t}, C_i\}$ and $y^s_{i+1,t} = \min\{C_{i+1}, w_{i+1}(\rho_{\max,i+1} - \rho_{i+1,t})\}$, where $y^d_{i,t}$ denotes the demand function corresponding to the maximum outflow from cell i at the time step t, and $y^s_{i+1,t}$ is the supply function corresponding to the maximum flow received by cell $i+1$ at the same time step t. Note that variables v_i, C_i, w_i, $\rho_{\max,i}$, ρ^c_i represent the model parameters of CTM, which can be calibrated using collected data by loop detector (Dervisoglu et al. 2009). Based on the results (Gomes and Horowitz 2006; Lo 1999), Equation (2) is reformulated as a linear programming problem, which is the key of the optimization model presented in the next section.

3. Optimization model

In this section, we firstly review the CTM-based deterministic optimization model adopted by Gomes and Horowitz (2006); Chow and Li (2014); Lo (1999); Ziliaskopoulos (2000), and then rewrite it as a distributionally robust chance constrained problem with consideration of the uncertain demand flows.

3.1. Deterministic optimization model

We rewrite the CTM-based freeway optimization model as follows:

$$\text{(DOM)} \min_r D = \sum_{i=1}^{I}\sum_{t=1}^{T}\left(\rho_{i,t}\Delta x_i\Delta t - \frac{f_{i,t}\Delta x_i\Delta t}{v_i}\right) + \sum_{j=1}^{J}\sum_{t=1}^{T} q_{j,t}\Delta t \tag{3}$$

s.t.

$$\rho_{i,t+1} = \rho_{i,t} + \frac{\Delta t}{\Delta x_i} \times (f_{i-1,t} - f_{i,t} + r_{i,t} - s_{i,t}), \forall i, t \tag{4}$$

$$f_{i,t} \le v_i\rho_{i,t}, \forall i, t \tag{5}$$

$$f_{i,t} \le C_i, \forall i, t \tag{6}$$

$$f_{i,t} \le C_{i+1}, \forall i, t \tag{7}$$

$$f_{i,t} \le w_{i+1}(\rho_{max,i+1} - \rho_{i+1,t}), \forall i, t \tag{8}$$

$$q_{j,t+1} = q_{j,t} + (d_{j,t} - r_{j,t})\Delta t, \forall j, t \tag{9}$$

$$q_{j,t} \le q_{max,j}, \forall j, t \tag{10}$$

$$0 \le r_{j,t} \le r_{max,j}, \forall j, t \tag{11}$$

To seek the optimal ramp metering r, the optimization problem above is to minimize the total delay D of the system over cells $i = 1, 2, \ldots, I$ and time $t = 1, 2, \ldots, T$. The objective function D includes the total delay in both mainline and on-ramps, where $\rho_{i,t}\Delta x_i\Delta t - \frac{f_{i,t}\Delta x_i\Delta t}{v_i}$ represents the mainline delay in cell i at time t (see Chow and Li (2014)), and $q_{j,t}$ denotes the queue length on the on-ramp j at time t. The constraint set (Equations (4)–(8)) is equivalent to the CTM, as stated by Gomes and Horowitz (2006), Lo (1999) and Ziliaskopoulos (2000). Both constraints (5) and (6) specify the demand limitations when the flow is under the free flow condition, whereas the constraints (7) and (8) characterize the supply limitations when the flow is under congested condition. We assume that the exit flows $s_{i,t}$ is given throughout the paper. Constraint (9) characterizes the evolution of queues $q_{j,t}$ on the on-ramps $j = 1, 2, \ldots, J$ at time step t, where J denotes the total number of on-ramps. We use $d_{j,t}$ to denote the variable of the demand flow intending to enter the freeway from on-ramp j at time step t and $r_{j,t}$ to denote the actual incoming demand flow entering the freeway from on-ramp j at time step t. Constraint (10) is used to govern the maximum queue size on the on-ramps avoiding that the unacceptably long queue on the on-ramps will be adopted as an optimal solution. Finally, constraint (11) gives the lower and upper bounds of the control variable r. The optimal ramp metering r is used to obtain the optimal control, which can also be realized through the hard shoulder running as well as the mainline speed control (Li, Chow, and Cassel 2014).

3.2. Distributionally robust chance constrained optimization model

Before presenting the distributionally robust chance constrained optimization model, we summarize the state-of-the-art of works in solving chance constraints in Table 1. By comparison, we adopt the approach proposed by Zymler, Kuhn, and Rustem (2013) to approximate distributionally robust chance constraints because of its established theoretical analysis and computational efficiency.

The deterministic optimization model (DOM) can be extended to the distributionally robust chance constrained optimization model (DRCCOM) with consideration of uncertain demand flows. Since the optimization problem (3) is a minimization problem and the constraint (9) is the only constraint associated with the traffic demand flows on the source links (such as on-ramps), both constraints (9) and (10) can be rewritten as the following constraints:

$$q_{j,t+1} = q_{j,t} + (\tilde{d}_{j,t} - r_{j,t})\Delta t, \forall j, t \tag{12}$$

and

$$\mathbb{P}(q_{j,t} - q_{max,j} \leq 0) \geq \varepsilon_d, \forall j, t \tag{13}$$

where $\tilde{d}_{j,t}$ denotes the random demand flow variable on the on-ramp j at time step t and $\varepsilon_d \in (0, 1)$ is the confidence parameter. The violation of constraint (13) means that the waiting queue length is longer than the maximum queue on source links. Due to the fact that the mean and covariance of uncertain demand flows are given, the chance constraint (13) can be rewritten as follows:

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(q_{j,t} - q_{max,j} \leq 0) \geq \varepsilon_d, \forall j, t \tag{14}$$

where $\mathcal{P}$ is the set of all probability distributions.

Table 1. Methods for solving chance constraint.

Type	Handling technique	Disadvantage
Scenario approximation (Calafiore and Campi 2005)	Using the constraints for the scenario sample points to replace the chance constraints	Prohibitively time consuming
Generator-based approximation (Nemirovski and Shapiro 2006)	The chance constraints are approximated by CVaR inequalities	The approximated problem might be intractable
Chebyshev's Relaxation (Sun et al. 2017)	Relaxing the chance constraints by Chebyshev's inequality	Solution is too conservative
Robust chance constraints (Zymler, Kuhn, and Rustem 2013; Hanasusanto et al. 2015)	Replacing the chance constraints by the Worst-case chance constraints under moment information, which can be transformed into a convex optimization problem with SDP or conic constraints	High computational cost

By Equation (4), the density $\rho_{i,t}$ can be rewritten as

$$\rho_{i,t} = \rho_{i,1} + \sum_{l=0}^{t-1} \frac{\Delta t}{\Delta x_i}(f_{i-1,l} - f_{i,l} + r_{i,l} - s_{i,l}), \forall i, t \tag{15}$$

Similarly, the queue length $q_{j,t}$ can be reformulated as

$$q_{j,t} = q_{j,1} + \sum_{l=0}^{t-1}(\tilde{d}_{j,l} - r_{j,l})\Delta t \tag{16}$$

Let $\tilde{\mathbf{d}}_j = \{0, \tilde{d}_{j,1}, \tilde{d}_{j,2}, \ldots, \tilde{d}_{j,T-1}\}^\top \in \mathbb{R}^T$, and $B_{t-1} = \{\Delta t, \Delta t, \Delta t, \ldots, \Delta t, 0, \ldots, 0\}^\top \in \mathbb{R}^T$. By the above relationship (16), the relationships (12) and (13) can be simplified as follows:

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\left\{q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} + B_{t-1}^\top \tilde{\mathbf{d}}_j \le 0\right\} \ge \varepsilon_d, \forall j, t \tag{17}$$

Then, the robust ramp metering problem with uncertain demand can be rewritten as:

$$\begin{aligned}\text{(DRCCP)} \min_r D = &\sum_{i=1}^{I}\sum_{t=1}^{T}\left([\rho_{i,1} + \sum_{l=0}^{t-1}\frac{\Delta t}{\Delta x_i}(f_{i-1,l} - f_{i,l} + r_{i,l} - s_{i,l})]\Delta x_i \Delta t - \frac{f_{i,t}\Delta x_i \Delta t}{v_i}\right)\\ &+ \mathbb{E}\left[\sum_{j=1}^{J}\sum_{t=1}^{T}\left(q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t + B_{t-1}^\top \tilde{\mathbf{d}}_j\right)\Delta t\right]\end{aligned} \tag{18}$$

subject to constraints (6), (7), (11), (15), (17) and

$$f_{i,t} \le v_i[\rho_{i,1} + \sum_{l=0}^{t-1}\frac{\Delta t}{\Delta x_i}(f_{i-1,l} - f_{i,l} + r_{i,l} - s_{i,l})], \forall i, t \tag{19}$$

$$f_{i,t} \le w_{i+1}\left[\rho_{max,i+1} - \rho_{i+1,1} - \sum_{l=0}^{t-1}\frac{\Delta t}{\Delta x_i}(f_{i,l} - f_{i+1,l} + r_{i+1,l} - s_{i+1,l})\right], \forall i, t \tag{20}$$

Due to the distributionally robust chance constraint (17), we have difficulty to solve the Problem (DRCCP) directly. Thus, we need to transform the problem into a solvable problem, which is presented in the next section.

4. Approximation of distributionally robust chance constraint

An approximation approach proposed by Zymler, Kuhn, and Rustem (2013) is utilized to approximate the constraint (17) in this section.

We let $\mu_j \in \mathbb{R}^T$ be the mean vector and $\Sigma_j \in \mathbb{S}^T$ be the covariance matrix of the random demand flow vector $\tilde{\mathbf{d}}_j$ under true distribution $\mathbb{P}$ throughout this paper. Thus, we implicitly assume that $\mathbb{P}$ has finite second-order moments. Without loss of generality, we assume that $\Sigma_j \succ 0$. To simplify the notation, we let

$$\Omega_j = \begin{bmatrix}\Sigma_j + \mu_j^\top \mu_j & \mu_j^\top \\ \mu_j & 1\end{bmatrix} \tag{21}$$

denote the second-order moment matrix of $\tilde{\mathbf{d}}_j$. Chen et al. (2010) has proved that the constraint (17) can be approximated by the Worst-Case CVaR constraint. Thus, we have

$$R(\alpha_{j,t}) = \left\{(f, r) : \sup_{\mathbb{P}\in\mathcal{P}} \text{CVaR}_{1-\varepsilon_d}\left(q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} + B_{t-1}^\top \tilde{\mathbf{d}}_j\right) \le 0\right\}, \forall j, t \tag{22}$$

where

$$\mathrm{CVaR}_{1-\varepsilon_d}\left(q_{j,1}-\sum_{l=0}^{t-1} r_{j,l}\Delta t-q_{max,j}+B_{t-1}^{\top}\tilde{\mathbf{d}}_j\right)$$
$$=\inf_{\alpha_{j,t}\in\mathbb{R}}\left\{\alpha_{j,t}+\frac{1}{1-\varepsilon_d}\mathbb{E}_{\mathbb{P}}\left[(q_{j,1}-\sum_{l=0}^{t-1} r_{j,l}\Delta t-q_{max,j}+B_{t-1}^{\top}\tilde{\mathbf{d}}_j-\alpha_{j,t})^{+}\right]\right\},\forall j,t \tag{23}$$

where $\alpha_{j,t}$ is the decision variable in chance constraints, $\mathbb{E}_{\mathbb{P}}(\bullet)$ denotes the expectation of the distribution $\mathbb{P}$, and $(\bullet)^{+}=\max\bullet, 0$ (see Rockafellar and Uryasev (2000, 2002) for more details).

An approximation approach proposed by Zymler, Kuhn, and Rustem (2013) based on the semidefinite programming (SDP) is utilized to approximate the constraint (22). By supposing that the mean and covariance matrix of stochastic variables are available, Zymler, Kuhn, and Rustem (2013) firstly used the Worst-case Conditional Value-at-Risk (WCVaR) constraints to approximate distributionally robust chance constraints, and then the WCVaR constraints were reformulated into SDP constraints. The results indicated that the approximation is exact when the roust individual chance constraint is a concave or quadratic function. In this paper, the approximation approach proposed by Zymler, Kuhn, and Rustem (2013) is adopted to approximate the constraint (17) and the equivalent form of constraint (22) is presented as the following theorem.

Theorem 4.1: *If the demand flow $\tilde{\mathbf{d}}_j$ follows an unknown probability distribution with given mean μ_j and covariance matrix Σ_j, then the constraint* (17) *can be approximated as follows:*

$$R(\alpha_{j,t})=\left\{(f,r):\begin{array}{c}\exists(\alpha_{j,t},\boldsymbol{A}_{j,t})\in\mathbb{R}\times\mathbb{S}^{T+1}\\ \alpha_{j,t}+\dfrac{1}{1-\varepsilon_d}\left\langle\Omega_j,\boldsymbol{A}_{j,t}\right\rangle\le 0,\boldsymbol{A}_{j,t}\ge\mathbf{0}\\ \boldsymbol{A}_{j,t}-\begin{bmatrix}\mathbf{0} & \dfrac{B_{t-1}}{2}\\ \dfrac{B_{t-1}^{\top}}{2} & q_{j,1}-\sum_{l=0}^{t-1} r_{j,l}\Delta t-q_{max,j}-\alpha_{j,t}\end{bmatrix}\ge\mathbf{0}\end{array},\forall j,t\right\}$$

where $\boldsymbol{A}_{j,t}\in\mathbb{S}^{T+1}$ is the $T+1$-dimensional real symmetric matrices, $\langle\Omega_j,\mathbf{A}_{j,t}\rangle=$ trace $(\Omega_j,\mathbf{A}_{j,t})$, which denotes a trace scalar product of matrices Ω_j and $\mathbf{A}_{j,t}$, and $\mathbf{A}_{j,t}\succcurlyeq\mathbf{0}$ implies that the matrix $\mathbf{A}_{j,t}$ is semidefinite.

Proof of Theorem 4.1:: It is noted that the constraint (22) can be equivalently expressed as $J(f, r,\alpha_{j,t})\le 0$, where

$$J(f,r,\alpha_{j,t})=\sup_{\mathbb{P}\in\mathcal{P}}\mathrm{CVaR}_{1-\varepsilon_d}\left(q_{j,1}-\sum_{l=0}^{t-1} r_{j,l}\Delta t-q_{max,j}+B_{t-1}^{\top}\tilde{\mathbf{d}}_j\right)$$
$$=\sup_{\mathbb{P}\in\mathcal{P}}\inf_{\alpha_{j,t}\in\mathbb{R}}\left\{\alpha_{j,t}+\frac{1}{1-\varepsilon_d}\mathbb{E}_{\mathbb{P}}\left[\left(q_{j,1}-\sum_{l=0}^{t-1} r_{j,l}\Delta t-q_{max,j}+B_{t-1}^{\top}\tilde{\mathbf{d}}_j-\alpha_{j,t}\right)^{+}\right]\right\} \tag{24}$$

By the stochastic saddle point theorem (Shapiro and Kleywegt 2002), the maximization and minimization operations can be interchanged as follows:

$$J(f,r,\alpha_{j,t})=\inf_{\alpha_{j,t}\in\mathbb{R}}\left\{\alpha_{j,t}+\frac{1}{1-\varepsilon_d}\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\left[(q_{j,1}-\sum_{l=0}^{t-1} r_{j,l}\Delta t-q_{max,j}+B_{t-1}^{\top}\tilde{\mathbf{d}}_j-\alpha_{j,t})^{+}\right]\right\} \tag{25}$$

Next, an SDP reformulation of the following worst-case expectation problem can be de-rived:

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[(q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} + B_{t-1}^{\top}\tilde{\mathbf{d}}_j - \alpha_{j,t})^{+}\right] \tag{26}$$

which can be regarded as the subordinate maximization problem in (25). Based on the Lemma (Zymler, Kuhn, and Rustem 2013), we obtain

$$\begin{aligned} &\inf_{\mathbf{A}_{j,t}\in\mathbb{S}^{T+1}} \langle \Omega_j, \mathbf{A}_{j,t}\rangle \\ &\text{s.t.} \quad \mathbf{A}_{j,t} \succeq \mathbf{0} \\ &[\tilde{\mathbf{d}}_j^{\top}\ 1]\mathbf{A}_{j,t}[\tilde{\mathbf{d}}_j^{\top}\ 1]^{\top} \geq q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} + B_{t-1}^{\top}\tilde{\mathbf{d}}_j - \alpha_{j,t}, \forall \tilde{\mathbf{d}}_j \in \mathbb{R}^T, j \in J, t \in T \end{aligned} \tag{27}$$

The constraint (27) can be written as follows:

$$\left[\tilde{\mathbf{d}}_j^{\top}\ 1\right]\mathbf{A}_{j,t}\left[\tilde{\mathbf{d}}_j^{\top}\ 1\right]^{\top} \geq q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} + B_{t-1}^{\top}\tilde{\mathbf{d}}_j - \alpha_{j,t}, \forall \tilde{\mathbf{d}}_j \in \mathbb{R}^T, j \in J, t \in T \tag{28}$$

Furthermore, constraint (28) can be equivalently expressed as

$$\mathbf{A}_{j,t} - \begin{bmatrix} \mathbf{0} & \dfrac{B_{t-1}}{2} \\ \dfrac{B_{t-1}^{\top}}{2} & q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} - \alpha_{j,t} \end{bmatrix} \succeq \mathbf{0}, \forall j, t$$

Therefore, the worst-case expectation problem (26) can be reformulated into

$$\begin{aligned} &\inf_{\mathbf{A}_{j,t}\in\mathbb{S}^{T+1}} \langle \Omega_j, \mathbf{A}_{j,t}\rangle \\ &\text{s.t.} \quad \mathbf{A}_{j,t} \succeq \mathbf{0} \\ &\mathbf{A}_{j,t} - \begin{bmatrix} \mathbf{0} & \dfrac{B_{t-1}}{2} \\ \dfrac{B_{t-1}^{\top}}{2} & q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} - \alpha_{j,t} \end{bmatrix} \succeq \mathbf{0}, \forall j, t \end{aligned} \tag{30}$$

Substituting (30) into (25) yields

$$J(f, r, \alpha_{j,t}) = \inf_{\alpha_{j,t}\in\mathbb{R}} \alpha_{j,t} + \frac{1}{1-\varepsilon_d}\langle \Omega_j, \mathbf{A}_{j,t}\rangle$$

$$\begin{aligned} &\text{s.t.} \quad \mathbf{A}_{j,t} \in \mathbb{S}^{T+1}, \mathbf{A}_{j,t} \succeq \mathbf{0} \\ &\mathbf{A}_{j,t} - \begin{bmatrix} \mathbf{0} & \dfrac{B_{t-1}}{2} \\ \dfrac{B_{t-1}^{\top}}{2} & q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} - \alpha_{j,t} \end{bmatrix} \succeq \mathbf{0}, \forall j, t \end{aligned} \tag{31}$$

and thus, the proof of Theorem 4.1 is completed. ■

Based on Theorem 4.1, the optimization problem (DRCCP) can be reformulated into a problem with SDP constraints as follows:

$$(\text{DRCCOM}) \min_{f,r,\alpha,\mathbf{A}} D = \sum_{i=1}^{I}\sum_{t=1}^{T}\left(\left[\rho_{i,1} + \sum_{l=0}^{t-1}\frac{\Delta t}{\Delta x_i}(f_{i-1,l} - f_{i,l} + r_{i,l} - s_{i,l})\right]\Delta x_i \Delta t - \frac{f_{i,t}\Delta x_i \Delta t}{v_i}\right)$$
$$+ \mathbb{E}\left[\sum_{j=1}^{J}\sum_{t=1}^{T}\left(q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t + B_{t-1}^{\top}\tilde{\mathbf{d}}_j\right)\Delta t\right] \tag{32}$$

subject to constraints (6), (7), (11), (15), (19), (20) and

$$\alpha_{j,t} + \frac{1}{1-\varepsilon_d}(\Omega_j, \mathbf{A}_{j,t}) \le 0, \forall j, t \tag{33}$$

$$\mathbf{A}_{j,t} - \begin{bmatrix} \mathbf{0} & \frac{B_{t-1}}{2} \\ \frac{B_{t-1}^{\top}}{2} & q_{j,1} - \sum_{l=0}^{t-1} r_{j,l}\Delta t - q_{max,j} - \alpha_{j,t} \end{bmatrix} \succeq \mathbf{0}, \forall j, t \tag{34}$$

$$\mathbf{A}_{j,t} \succeq \mathbf{0}, \mathbf{A}_{j,t} \in \mathbb{S}^{T+1}, \forall j, t \tag{35}$$

Based on the result (Zymler, Kuhn, and Rustem 2013), we get that DRCCOM can be solved efficiently.

5. Case study

We select a 13 km road of the Kwinana Freeway in the vicinity of Perth in Australia (refer to Figure 2). We discretize the section into 26 cells with 500 m for each cell. The road section covers eight on-ramps and four off-ramps, and is one of the busiest sections in Perth. We select the duration from 6: 00am to 10: 00am, which represents the peak hours. The on-ramps are located at cells 2, 5, 8, 9, 10, 16, 17, 25 and the off-ramps are located at cells 3, 7, 15, 26. The data of flow and density collected from loop detectors are used to obtain the piecewise linear fundamental diagram for each cell. The associated parameters are listed in Table 2.

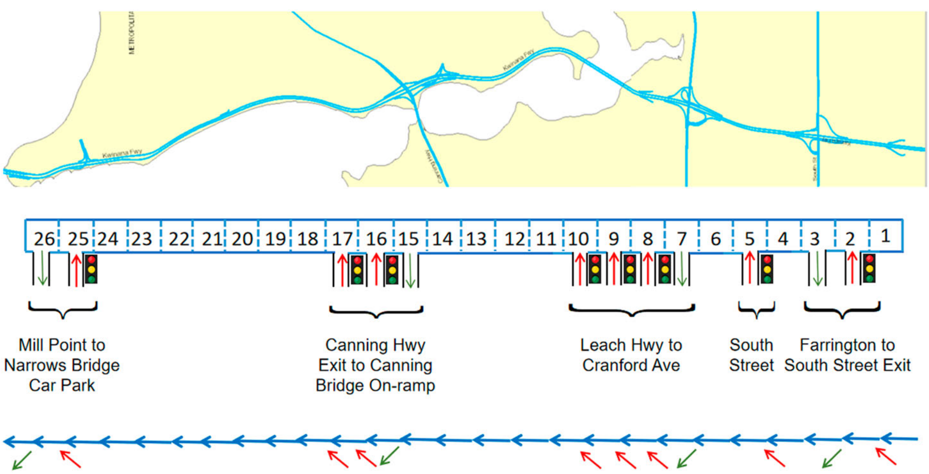

Figure 2. Road map.

Table 2. Model parameters.

Name of parameter	Value
$v_i, i = 1, \cdots, 24$	27.7778 m/s
$v_i, i = 25, 26$	22.2222 m/s
$w_i, i = 1, 3, 4, 6, \cdots, 24$	9.8029 m/s
$w_i, i = 2, 5$	9.7895 m/s
$w_i, i = 25, 26$	10.7830 m/s
$C_i, i = 1, 3, 4, 6, \cdots, 24$	1.6667 veh/s
$w_i, i = 2, 5, 25, 26$	2.2222 veh/s
$q_{max,j}, j = 7$	120 veh
$q_{max,j}, j = 1, \cdots, 6, 8$	60 veh
$r_{max,j}, j = 1, \cdots, 8$	0.5500 veh/s
$\rho_{max,i}, i = 1, 3, 4, 6, \cdots, 24$	0.2300 veh/m
$\rho_{max,i}, i = 2, 5, 25, 26$	0.3067 veh/m
Δt	15 s
$\Delta x_i, i = 1, \cdots, 26$	500 m

5.1. Deterministic model

The objective of the deterministic optimization model (DOM) is to find an optimal ramp metering by minimizing the total travel delay with consideration of deterministic demand flows under triangular fundamental diagrams. We use MATLAB R2019a with SeDuMi (Sturm 1999) solver and the YALMIP interface (Löfberg 2004) to solve the proposed models in the numerical application. For the no control model and DOM, we show the corresponding results on 11 June and 13 June 2018, in Figures 3 and 4, respectively. The bar on the right side shows the size of mainline density that increases from bottom to top. The lighter colour in the figures implies smaller value of density and better traffic condition and vis versa. For the no control model on 11 June 2018, the total system delay with zero ramp delay is 1074.6552veh-hr and the total system delay for DOM is 488.8045veh-hr and the associated ramp delay is 174.8159veh-hr. On 13 June 2018, the total system delay for no control with zero ramp delay is 1583.9869veh-hr and the total system delay for DOM is 1105.5529veh-hr and the associated ramp delay is 474.3417veh-hr. By comparison, we can see that there are 54.5152% and 30.2045% improvement for the total delay, respectively. Therefore, ramp metering can reduce the congestion on the freeway, especially for the sections with on-ramps, and hence ramp metering is an effective strategy to improve freeways operations.

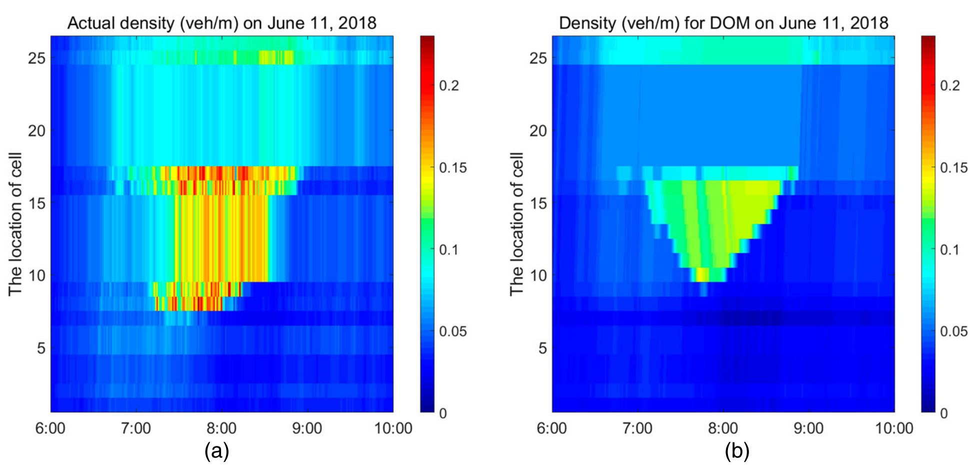

Figure 3. Density (veh/m) for no control and DOM on 11 June 2018.

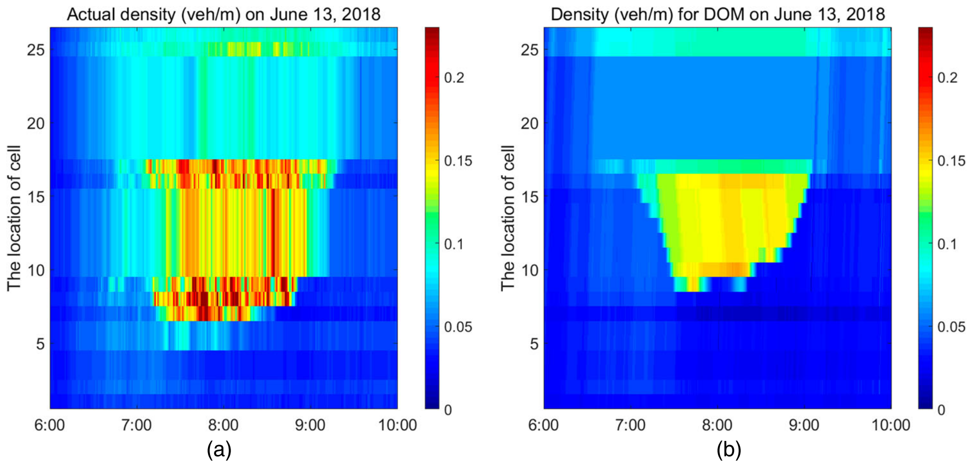

Figure 4. Density (veh/m) for no control and DOM on 13 June 2018.

5.2. Disributionally robust chance constrained optimization model

Now, we show the corresponding results of the disributionally robust chance constrained optimization model (DRCCOM), where we take into account ten scenarios (refer to Table 3 for more details) based on the measured data of the demand flows. On the second column in Table 3, '1.00' denotes the situation where the demand flow $\tilde{d}_{j,t}$ is set as the mean demand flow $d_{j,t}^{mean}$; '0.96' denotes the situation where the demand flow $\tilde{d}_{j,t}$ is reduced to $0.96d_{j,t}^{mean}$; '1.05' refers to the situation in which the demand flow $\tilde{d}_{j,t}$ is increased to $1.05d_{j,t}^{mean}$. A demand multiplier more than 1 refers to the situation where the actual demand flow is being underestimated and vis versa. Figure 5 shows the mean mainline demand, mean on-ramp demands and mean out-going flows.

For comparison, the robust optimization method (ROM) proposed by Chow and Li (2014) is adopted in this paper, where we suppose that there is an uncertainty of ± 0.05 associated with the demand flows based on collected data from detectors and utilize triangular fundamental diagrams. This gives $\tilde{d}_{j,t}^{min} = 0.95 * \tilde{d}_{j,t}$ and $\tilde{d}_{j,t}^{max} = 1.05 * \tilde{d}_{j,t}$. First, we take into account the scenario 5, i.e. letting $\tilde{d}_{j,t} = d_{j,t}^{mean}$. Table 4 shows the corresponding results. By analyzing, we can see that the performance of DRCCOM is the best when $\varepsilon_d = 0.95$. For this case, the total delay and ramp delay for the DRCCOM are 731.6660veh-hr and 215.0415veh-hr, respectively. Compared to ROM, there are, respectively, 0.2442% and 0.0670% improvement. Compared with DOM, the total delay of DRCCOM is less, but the ramp delay is more than that of DOM. The performances of total delay of DRCCOM when $\varepsilon_d = 0.90$ or $\varepsilon_d = 0.97$ do not outperform those of DOM and ROM. Furthermore, we can see that the ramp delay is

Table 3. Ten scenarios based on the measured data of the demand flows.

Case	Demand flows	Control strategy
1	$0.96 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
2	$0.97 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
3	$0.98 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
4	$0.99 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
5	$1.00 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
6	$1.01 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
7	$1.02 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
8	$1.03 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
9	$1.04 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust
10	$1.05 * \tilde{d}_{j,t}$	Deterministic, robust, distributionally robust

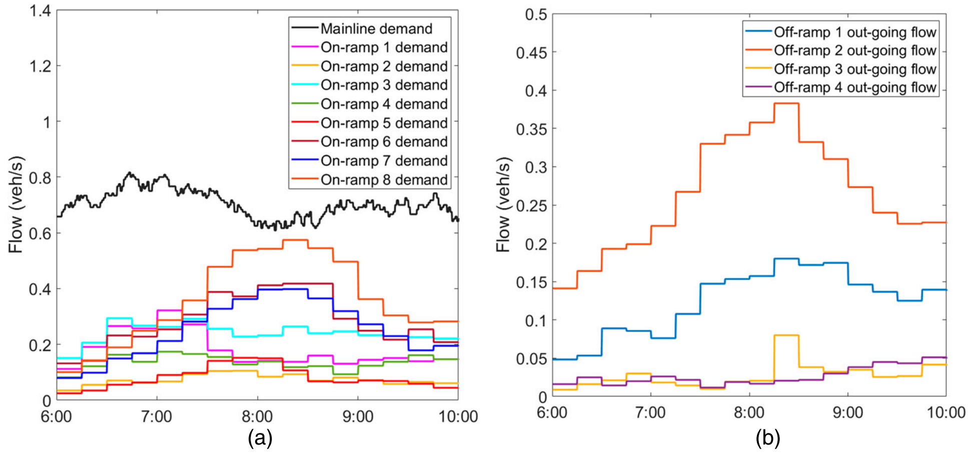

Figure 5. Mean mainline demand, mean on-ramp demands and mean out-going flows.

Table 4. Total delay and ramp delay for ten scenarios.

Demand	DOM	ROM	DRCCOM with $\varepsilon_d = 0.90$	DRCCOM with $\varepsilon_d = 0.95$	DRCCOM with $\varepsilon_d = 0.97$
	Total delay (veh-hr) Ramp delay (veh-hr)	Total delay (veh-hr) Ramp delay (veh-hr)	Total delay (veh-hr) Ramp delay (veh-hr)	Total delay (veh-hr) Ramp delay (veh-hr)	Total delay (veh-hr) Ramp delay (veh-hr)
$0.96{*}d_{j,t}^{mean}$	332.6767	338.9177	334.7076	328.2172	329.8285
	40.0145	48.1446	48.5633	45.3767	42.9587
$0.97{*}d_{j,t}^{mean}$	411.3381	420.2148	417.3771	410.2358	413.6383
	82.0282	94.3712	89.4254	85.7465	83.1289
$0.98{*}d_{j,t}^{mean}$	502.4933	500.5120	503.7418	498.2489	500.9168
	102.9199	102.9337	110.8222	105.0563	102.9788
$0.99{*}d_{j,t}^{mean}$	599.4085	602.6364	605.5120	598.6296	603.5045
	185.6659	208.0696	202.9336	196.4971	186.2164
$1.00{*}d_{j,t}^{mean}$	734.4483	733.4570	736.4828	731.6660	733.4828
	213.7042	216.4920	218.5041	215.0415	214.7027
$1.01{*}d_{j,t}^{mean}$	909.7404	906.5402	915.7383	896.2559	907.6203
	346.3851	357.6635	366.2991	356.7829	350.5417
$1.02{*}d_{j,t}^{mean}$	1174.9983	1081.2249	1168.0312	1084.1379	1164.1543
	397.6578	446.8358	429.1136	418.0353	408.7932
$1.03{*}d_{j,t}^{mean}$	1338.1348	1336.3029	1344.1208	1304.6056	1321.0637
	503.1153	564.3454	592.6797	551.3854	529.0421
$1.04{*}d_{j,t}^{mean}$	1569.6749	1521.8225	1581.2552	1555.3102	1571.8093
	629.6253	659.5434	663.7006	658.4050	646.6241
$1.05{*}d_{j,t}^{mean}$	1845.0960	1805.6598	1838.0398	1798.5056	1824.0285
	727.3990	725.9099	787.2781	765.9304	743.8254

increasing with the value of ε_d decreasing, which is consistent with our theoretical analyses because smaller ε_d implies more probability of violation of constraint in terms of that the waiting queue length is longer than the maximum queue on an on-ramp.

Table 4 also shows the total delay and ramp delay over 10 scenarios for the three different control strategies. We can see that the total delay and ramp delay increase with the demand increasing for the three control strategies. In particular, the total delay of DRCCOM when $\varepsilon_d = 0.95$ is less than those of ROM for the cases that $\tilde{d}_{j,t} \neq 1.02 * d_{j,t}^{mean}, 1.04 * d_{j,t}^{mean}$. Compared to DOM, the total delay of DRCCOM when $\varepsilon_d = 0.95$ is less than those of DOM for all the 10 different scenarios, while the ramp delay of DRCCOM shows an opposite trend. Moreover, the performances of total delay and ramp delay of DOM are better than those of DRCCOM when $\varepsilon_d = 0.90$ for most cases. When $\varepsilon_d = 0.97$, the total delay of DRCCOM is less than those of DOM for most cases, but the ramp delay is longer than DOM. This is reasonable because the objective function is to minimize total delay rather than ramp delay. When more vehicles are holding on on-ramps, total delay may reduce but ramp delay will increase.

6. Conclusion

A distributionally robust chance constrained optimization model is presented in this paper to address the ramp metering problem with uncertain demand flows. The model is formulated as a semidefinite programming using the Worst-Case Conditional Value-at-Risk (WCVaR) constraints to approximate distributionally robust chance constraints. Given partial information (such as mean and covariance matrix) of stochastic demand flows and the triangular fundamental diagrams, an optimal ramp metering strategy can be obtained by minimizing the total delay of mainline and on-ramps. The associated results of three different ramp metering strategies, i.e. deterministic optimization model, robust optimization model and distributionally robust chance constrained optimization model, are presented and compared. The results show that, considering the uncertainty encountered by the system over a range of scenarios, distributionally robust optimization is an effective and useful method to control the total delay of the system and mitigate traffic congestions. The results also show that the distributionally robust optimization is more effective in managing the total system delay. Further studies will take into account the distributionally robust optimization model under set-valued fundamental diagrams and the route choice behaviour of drivers.

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